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WAVEFORMS AND AMBIGUITY FUNCTIONS OF PULSED SIGNALS REFLECTED FROM A SPHERICAL SATELLITE

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REPORT
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ABSTRACT

This paper deals with the calculation of the waveforms which are reflected from a spherical satellite surface, and with their ambiguity functions. These studies will be helpful in radar measurements of the surface characteristics of a satellite.

A waveform consisting of a finite pulse train repeated periodically has an ambiguity function which is periodic along both axes, i.e., doppler shift and delay time. The effect of this on measurements of the surface characteristics of a satellite is discussed.

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WAVE FORMS AND AMBIGUITY FUNCTIONS OF PULSED SIGNALS REFLECTED FROM A SPHERICAL SATELLITE

HYPOTHESES

1. A satellite is rotating around the z axis with an angular velocity Ω.

2. Incident and reflected waves propagate along the x axis for the assumed monostatic scattering

3. The transmitted signal is a pulse train with uniform shape and spacing and is repeated periodically.

NOTATION

R = radius of a satellite
$e_i(t)$ = incident wave
$e_r(t)$ = reflected wave
c = velocity of light
f_o = carrier frequency of the transmitted wave
ΔT = a time dalay in the reflected wave from a point $P(\theta, \phi)$ on the surface, with difference between a point $P(\theta, \phi)$ and the point which is nearest to the transmitter (see Fig. 1)

(1) $$\Delta T = \frac{2R}{c} (1 - \sin \theta \cdot \cos \phi)$$

Δf = a doppler shift in the reflected wave from a point $P(\theta, \phi)$

(2) $$\Delta f = \frac{2f_o}{c} \Omega R \sin \theta \cdot \sin \phi$$

$|X(\theta, \phi, f_d, T_d)|$ = an ambiguity function of "the matched filter output" which is designed to match the reflected wave from a point $P(\theta, \phi)$

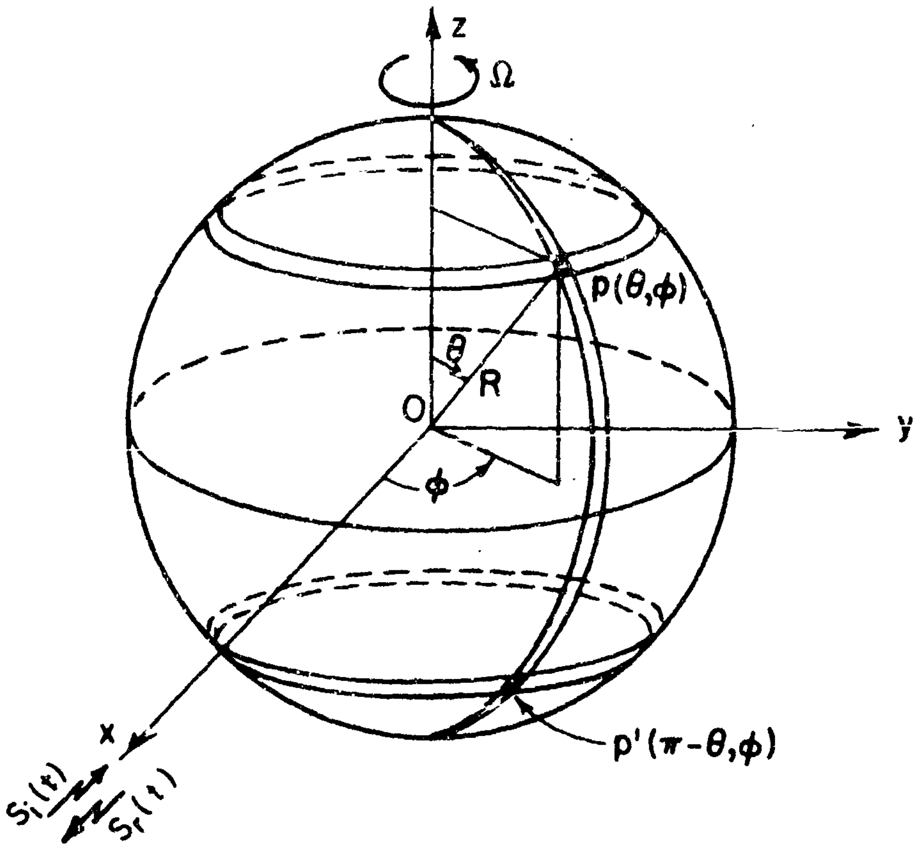


Fig. 1. A coordinate for a satellite.

$|X(\Delta f, \Delta T, f_d, T_d)|$ = another expression of an ambiguity function $|X(\theta, \phi, f_d, T_d)|$ by substituting Δf and ΔT for θ and ϕ.

REFLECTED WAVES AND THE MATCHED FILTERS

Let the transmitted wave be

$$(3) \qquad e_i(t) = U(t) \cdot \epsilon^{j2\pi f_o t} ,$$

where

U(t) is the envelope function of a transmitted wave and f_0 is the carrier frequency. The reflected wave from a point $P(\theta, \phi)$ on the surface is

$$(4) \qquad e_r(t) = \nu^{\frac{1}{2}}(\theta, \phi) R^2 \sin\theta \cdot \cos\phi \cdot U(t-\Delta T) \cdot \epsilon^{j2\pi(f_0 \pm \Delta f)(t-\Delta T)} ,$$

where

$\nu^{\frac{1}{2}}(\theta, \phi)$ is a scattering coefficient of the surface and depends on θ and ϕ, and the sign of $\pm\Delta f$ is $+\Delta f$ for $0 < \phi \leq \pi/2$ and $-\Delta f$ for $0 > \phi \geq -\pi/2$.

If $e_i(t)$ is assumed to be a pulse train, as shown in Fig. 2, its formula can be written

$$(5) \qquad e_i(t) = V(t) \cdot \sum_{n=-\infty}^{\infty} U(t-nT_s),$$

where V(t) is the envelope function of the pulse train and U(t) is the component pulse function of the same.

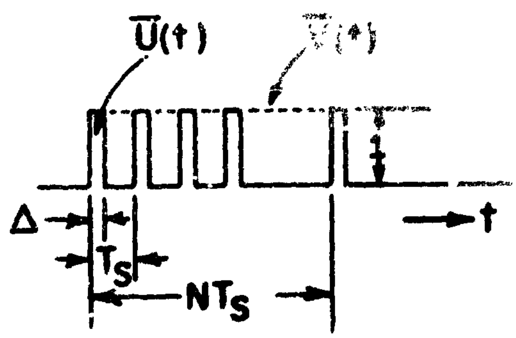


Fig. 2. A pulse train.

If the wave is periodic, as shown in Fig. 3, V(t) changes as

$$(6) \qquad V(t) = \sum_{i=-\infty}^{\infty} V(t - iT_\ell) = \sum_{i=-\infty}^{\infty} V(t - i \cdot \overline{N+M} \cdot T_s) \, .$$

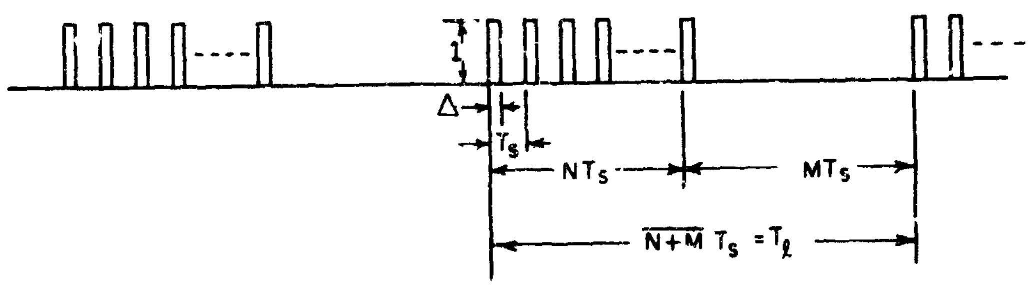


Fig. 3. Transmitted pulse train.

From Eqs. (4) and (5), the reflected wave from the satellite is

$$(7) \qquad e_r(t) = 2 \cdot \nu^{\frac{1}{2}}(\theta, \phi) \cdot R^2 \cdot \sin\theta \cdot \cos\phi \cdot V(t - \Delta T) \cdot \sum_{n=-\infty}^{\infty} U(t - nT_s' - \Delta T) \cdot \epsilon^{j2\pi(f_o \pm \Delta f)\cdot(t - \Delta T)} \, .$$

In Eq. (7), the factor 2 on the right of the equal sign indicates the contribution from a point $P'(\theta, \phi)$ on the other hemisphere of the surface; and the prime sign of T_s' indicates the effect of doppler shift in frequency which is equivalent to changing the scale unit to $\frac{1}{1 \pm \frac{\Delta f}{f_o}}$ from unity. T_s' is

$$(8) \qquad T_s' = \frac{T_s}{1 \pm \frac{\Delta f}{f_o}} = \frac{T_s}{1 \pm \frac{2}{c}\,\Omega R \sin\theta \cdot \sin\phi} \, .$$

Substituting Eqs. (1) and (2) into Eq. (7) we get a formula for the reflected wave as

$$(9) \qquad e_r(t) = 2 \cdot \nu^{\frac{1}{2}}(\theta,\phi) R^2 \sin\theta \cdot \cos\phi \cdot V\{t - \frac{2R}{c}(1 - \sin\theta \cdot \cos\phi)\}$$

$$\cdot \sum_{n=-\infty}^{\infty} U\{t - nT_s' - \frac{2R}{c}(1 - \sin\theta \cdot \cos\phi)\}$$

$$\cdot \epsilon^{j2\pi\{f_0 \pm \frac{2f_0}{c}\Omega R \sin\theta \cdot \sin\phi\}\{t - \frac{2R}{c}(1 - \sin\theta \cdot \cos\phi)\}} \quad .$$

Total reflection from the satellite is

$$(10) \qquad \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=-\frac{\pi}{2}}^{\frac{\pi}{2}} e_r(t) d\phi \, d\theta \quad .$$

To detect surface roughness on the satellite it is necessary to design matched filters corresponding to each point $P(\theta,\phi)$. Then, we may write the responses of these matched filters as $H(f,\theta,\phi)$ rather than $H(f)$. We can determine the frequency spectrum of the reflected wave by the Fourier transform of Eq. (9). Further, by omitting the carrier frequency component from the reflected signal, Eq. (9) becomes

$$(11) \qquad e_r(f) = A(\theta,\phi) \cdot U(f) \cdot \sum_{n=-\infty}^{\infty} V(f - n\frac{1}{T_s'}) \quad ,$$

where

$$(12) \qquad A(\theta,\phi) = 2 \cdot \nu^{\frac{1}{2}}(\theta,\phi) \cdot R^2 \sin\theta \cdot \cos\phi, \text{ and}$$

$$(13) \qquad U(f) = \int_{-\infty}^{\infty} U(t - \Delta T)\, \epsilon^{-j2\pi ft} dt = U'(f)\, \epsilon^{-j2\pi f \Delta T} \quad ,$$

$$U'(f) = 2\int_{-\frac{\Delta'}{2}}^{\frac{\Delta'}{2}} \epsilon^{-j2\pi ft} dt = 2\,\frac{\epsilon^{-j2\pi f\frac{\Delta'}{2}} - \epsilon^{+j2\pi f\frac{\Delta'}{2}}}{-j2\pi f} = 2\Delta' \frac{\sin \pi\Delta' f}{\pi\Delta' f} \ .$$

$U'(f)$ is the frequency spectrum of a component pulse and Δ' is its width with doppler shift;

$$(14) \qquad \frac{\Delta'}{T_s'} = \frac{\Delta\left(1 \pm \frac{\Delta f}{f_o}\right)^{-1}}{T_s\left(1 \pm \frac{\Delta f}{f_o}\right)^{-1}} = \frac{\Delta}{T_s} = \beta_s \ .$$

Equation (14) shows that the duty cycle of the transmitted pulse is equal to that of the reflected wave.

If the envelope $V(t)$ is periodic, as seen in Fig. 3, we can rewrite Eq. (11) from Eqs. (5) and (6) as

$$(15) \qquad e_r(t) = A(\theta,\phi) \cdot \sum_{i=-\infty}^{\infty} V(t-i\,\overline{N+M}\,T_s') \sum_{n=-\infty}^{\infty} U(t-nT_s') \ ,$$

or

$$= A(\theta,\phi) \left[V(t) * \sum_{i=-\infty}^{\infty} \delta(t-i\,\overline{N+M}\,T_s') \right] \cdot \left[U(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT_s') \right] ,$$

where * indicates convolution and $\delta(t)$ is the impulse function.

Fourier transform of Eq. (15) shows the frequency spectrum of the reflected wave, then, as

$$(16) \qquad e_r(f) = A(\theta,\phi) \left[\frac{1}{\overline{N+M}\,T_s'} V(f) \sum_{i=-\infty}^{\infty} \delta\left(f - \frac{i}{\overline{N+M}\,T_s'}\right) \right] *$$

$$\left[\frac{1}{T_s'}\, U(f) \sum_{n=-\infty}^{\infty} \delta\left(f-n\,\frac{1}{T_s'}\right)\right],$$

or

$$= A(\theta,\phi)\left[\frac{1}{\overline{N+M}\; T_s'} \sum_{i=-\infty}^{\infty} V\left(f-\frac{i}{\overline{N+M}\; T_s'}\right)\right]$$

$$* \left[\frac{1}{T_s'} \sum_{n=-\infty}^{\infty} U\left(f-n\,\frac{1}{T_s'}\right)\right] \quad ,$$

where

$$(17) \qquad V\left(f-\frac{i}{\overline{N+M}\; T_s'}\right) = 2\int_{-\frac{NT_s'}{2}}^{\frac{NT_s'}{2}} V(t-\Delta T)\, \epsilon^{-j2\pi\left(f-\frac{i}{\overline{N+M}\; T_s'}\right)t}\, dt$$

$$= 2NT_s' \frac{\sin \pi NT_s'\left(f-\frac{i}{\overline{N+M}\; T_s'}\right)}{\pi NT_s'\left(f-\frac{i}{\overline{N+M}\; T_s'}\right)} \cdot \epsilon^{-j2\pi\left(f-\frac{i}{\overline{N+M}\; T_s'}\right)\Delta T} .$$

From Eq. (13)

$$(13') \qquad U\left(f-n\,\frac{1}{T_s'}\right) = 2\int_{-\frac{\Delta'}{2}}^{\frac{\Delta'}{2}} U(t-\Delta T)\, \epsilon^{-j2\pi\left(f-\frac{n}{T_s'}\right)t}\, dt$$

$$= 2\Delta' \frac{\sin \pi\Delta'\left(f-n\frac{1}{T_s'}\right)}{\pi\Delta'\left(f-n\,\frac{1}{T_s'}\right)}\, \epsilon^{-j2\pi\left(f-\frac{n}{T_s'}\right)\Delta T} .$$

By substituting Eqs. (17) and (13') into Eq. (16)

$$(18) \qquad e_r(f) = A(\theta,\varphi)2\beta_s \cdot 2\beta_\ell \left[\sum_{i=-\infty}^{\infty} \frac{\sin \pi N T_s' \left(f - \frac{i}{N+M\ T_s'}\right)}{\pi N T_s' \left(f - \frac{i}{N+M\ T_s'}\right)} \epsilon^{-j2\pi \left(f - \frac{i}{N+M\, T_s'}\right) \Delta T} \right] * \left[\sum_{n=-\infty}^{\infty} \frac{\sin \pi\Delta' \left(f - \frac{n}{T_s'}\right)}{\pi\Delta' \left(f - \frac{n}{T_s'}\right)} \epsilon^{-j2\pi \left(f - \frac{n}{T_s'}\right) \Delta T} \right],$$

where

$$\beta_s = \frac{\Delta'}{T_s'} = \frac{\Delta}{T_s}, \quad \beta_\ell = \frac{N T_s'}{N+M\ T_s'} = \frac{N}{N+M},$$

$$\Delta' = \frac{\Delta}{1 \pm \frac{\Delta f}{f_o}}, \qquad T_s' = \frac{T_s}{1 \pm \frac{\Delta f}{f_o}},$$

$$\Delta T = \frac{2R}{c}(1 - \sin\theta \cdot \cos\varphi), \text{ and } \Delta f = \frac{2f_o}{c}\Omega R \sin\theta \cdot \sin\phi .$$

In Figs. 4(a-g) the graphs of $e_r(f)$ are shown. It is easily understood that the origin of the coordinate corresponds to the carrier frequency with the doppler shift.

From the definition of the matched filter H(f)

$$(19) \qquad H(f) = G_a \widetilde{e_r(f)}\, \epsilon^{-j2\pi f t_1},$$

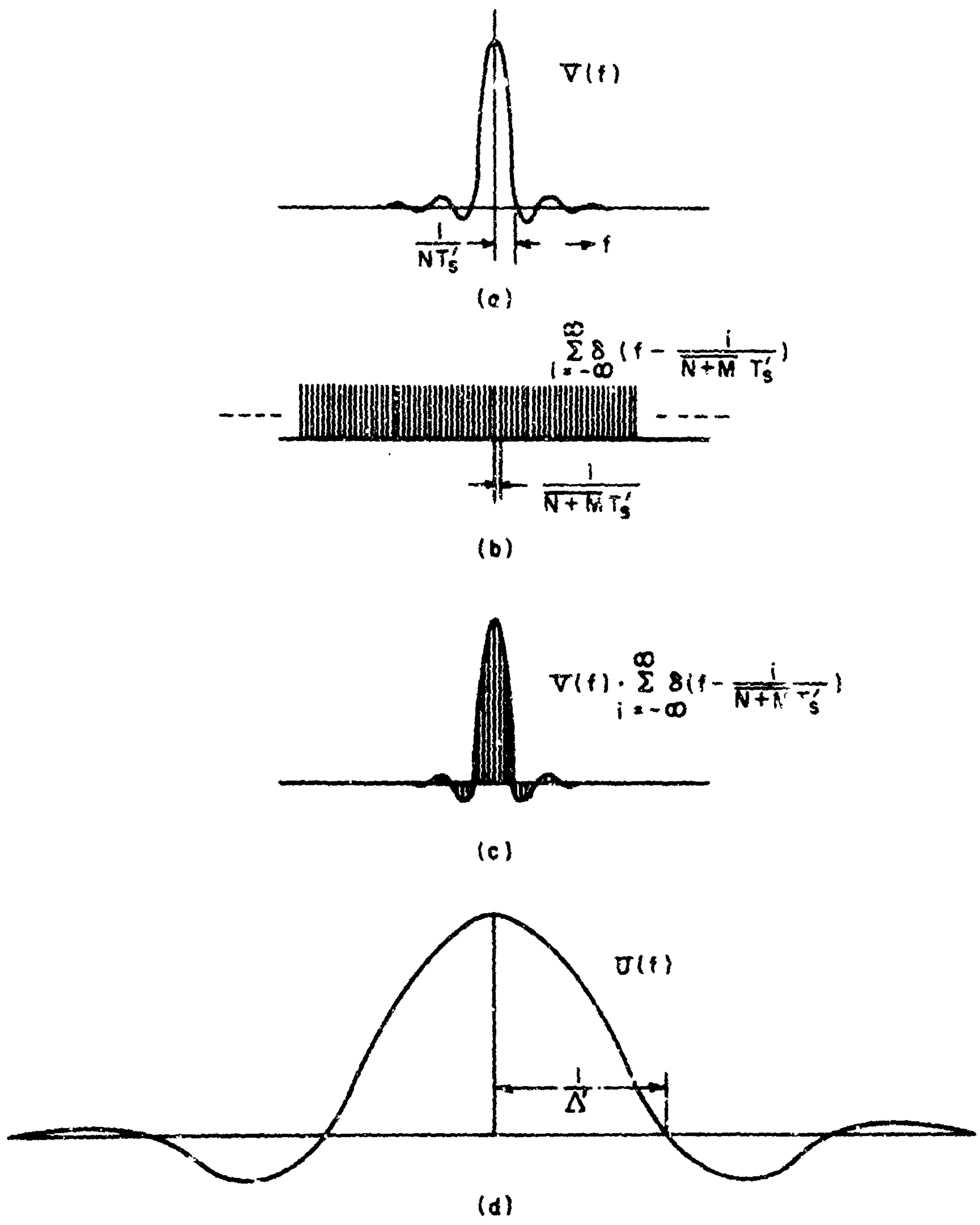


Fig. 4. Frequency spectrum of the reflected signal from a satellite.

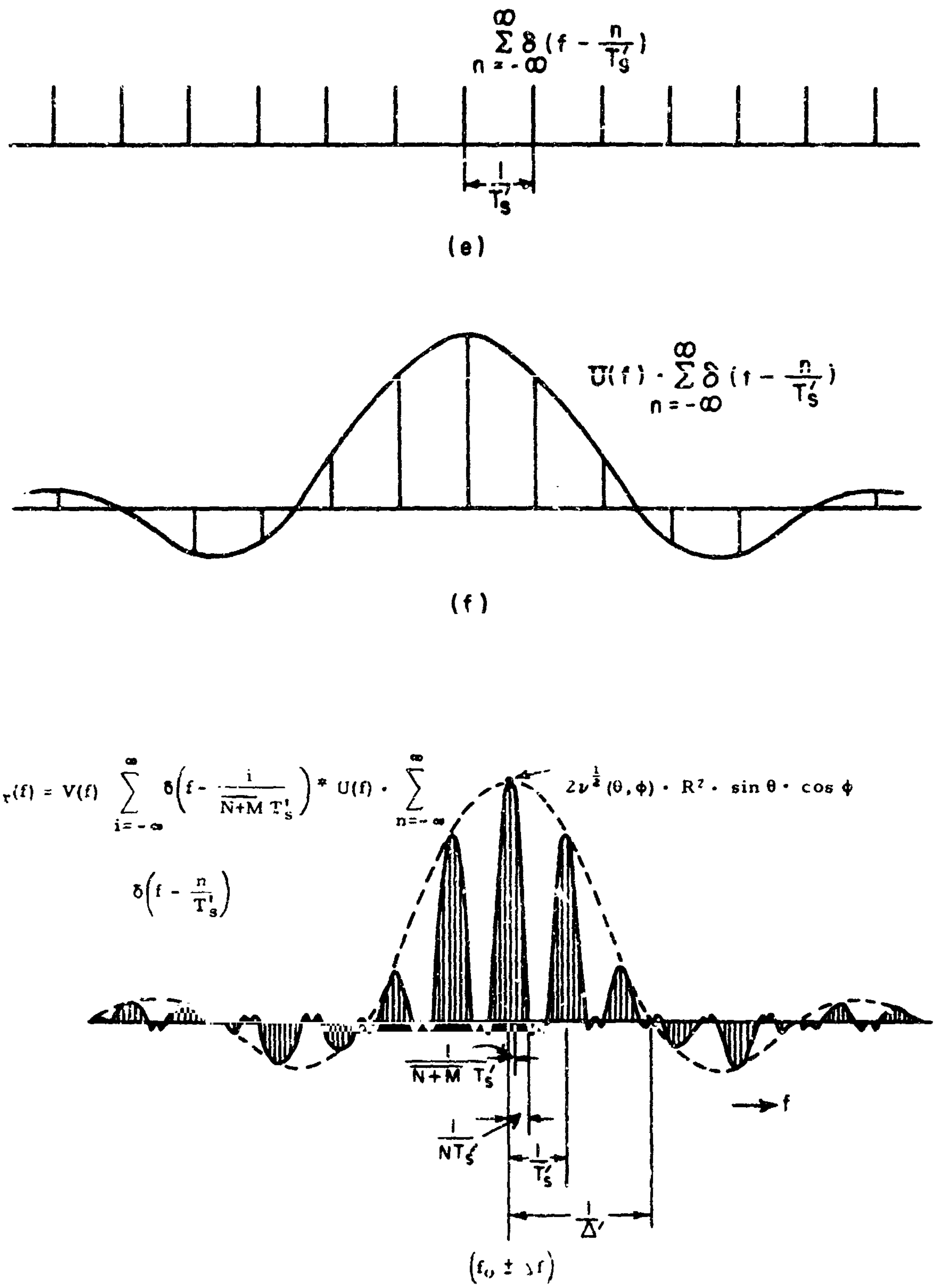


Fig. 4. Frequency spectrum of the reflected signal from a satellite.

where

G_a is a constant equal to the maximum filter gain and usually taken to be unity,

$\widetilde{e_r(f)}$ is a complex conjugate of $e_r(f)$, and

t_1 is a fixed value of time at which the signal is observed.

Combining Eqs. (18) and (19) we get $H(f,\theta,\phi)$ as

$$(20) \qquad H(f,\theta,\phi) = A(\theta,\phi) 2\beta_s \cdot 2\beta_\ell \left[\sum_{i=-\infty}^{\infty} \frac{\sin \pi N T_s'\left(f - \frac{i}{\overline{N+M}\, T_s'}\right)}{\pi N T_s' \left(f - \frac{i}{\overline{N+M}\, T_s'}\right)} \epsilon^{j2\pi\left(f - \frac{i}{\overline{N+M}\, T_s'}\right)\Delta T} \right] * \left[\sum_{n=-\infty}^{\infty} \frac{\sin \pi \Delta' \left(f - \frac{n}{T_s'}\right)}{\pi \Delta' \left(f - \frac{n}{T_s'}\right)} \epsilon^{j2\pi\left(f - \frac{n}{T_s'}\right)\Delta T} \right] \epsilon^{j2\pi f t_1} .$$

Impulse response of $H(f,\theta,\phi)$ is

$$(21) \qquad h(t,\theta,\phi) = \int_{-\infty}^{\infty} H(f)\, \epsilon^{j2\pi f t}\, df = e_r(t_1 - t) .$$

By substituting Eq. (15) into Eq. (21) and adding ΔT, Eq. (21) becomes

$$(22) \qquad h(t,\theta,\phi) = A(\theta,\phi) \left[V(t_1 + \Delta T - t) * \sum_{i=-\infty}^{\infty} \delta(t_1 + \Delta T - t + i\, \overline{N+M}\, T_s') \right] \cdot \left[U(T_1 + \Delta T - t) * \sum_{n=-\infty}^{\infty} \delta(t_1 + \Delta T - t + n T_s') \right] ,$$

or

$$= A(\theta,\phi)\left[\sum_{i=-\infty}^{\infty} V(t_1+\Delta T-t+i\,\overline{N+M}\;T_s')\right]$$

$$\cdot\left[\sum_{n=-\infty}^{\infty} U(t_1+\Delta T-t+nT_s')\right] .$$

The output of a matched filter in general is as shown

(23) $$e_o(t) = \int_{-\infty}^{\infty} e_r(f)\cdot H(f)\cdot \epsilon^{j2\pi ft}\,df$$

or

(23') $$e_o(t) = e_r(t) * h(t) .$$

Since it is desired to determine the surface characteristics of the satellite, it is necessary, at least conceptually, to arrange many matched filters corresponding to each of the divided surface portions as shown in Fig. 5. The output of a matched filted $H_{pq}(f)$ or $h_{pq}(t)$ resulting from a signal reflected from $\theta_{p'}, \phi_{q'}$ may be expressed as $e_o(f)$ or $e_o(t)$, i.e.,

(24) $$e_o(t)_{p-p',q-q'} = \int_{-\infty}^{\infty} e_r(f)_{p'q'}\cdot H(f)_{pq}\cdot \epsilon^{j2\pi ft}\,df$$

or

(24') $$e_o(t)_{p-p',q-q'} = e_r(t)_{p'q'} * h_{pq}(t) .$$

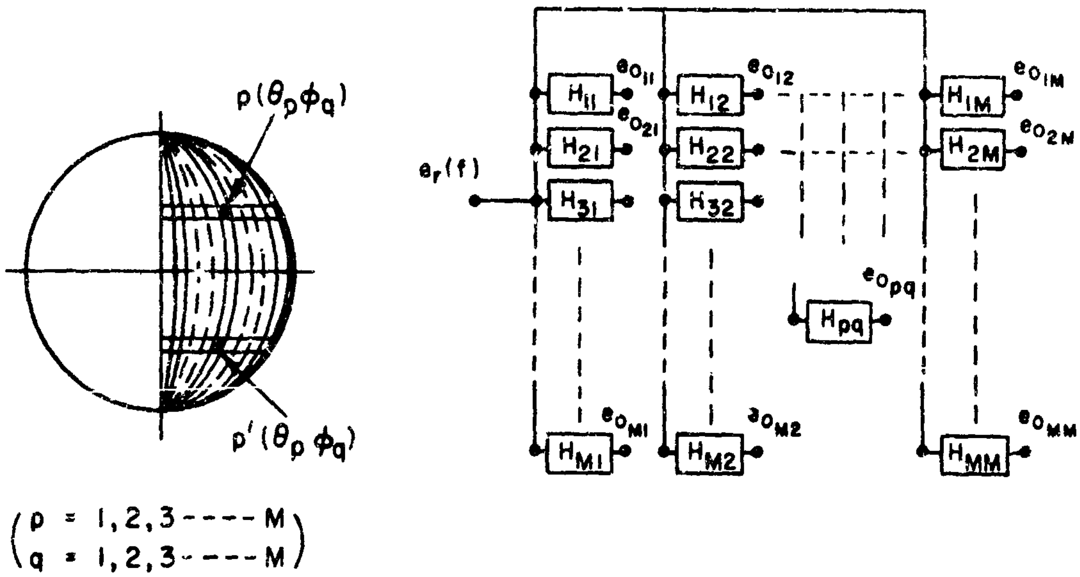


Fig. 5. Matched filters arrangement.

To calculate Eq. (24) or (24'), put ΔT_{pq} and Δf_{pq} into $H(f,\theta,\phi)$ or $h(t,\theta,\phi)$, and $\Delta T_{p'q'}$ and $\Delta f_{p'q'}$ into $e_r(f)$ or $e_r(t)$,

where

$$\Delta T_{pq} = \frac{2R}{c} (1 - \sin \theta_p \cdot \cos \varphi_q) ,$$

$$\Delta T_{p'q'} = \frac{2R}{c} (1 - \sin \theta_{p'} \cdot \cos \phi_{q'}) ,$$

$$\Delta f_{pq} = \frac{2f_o}{c} \Omega R \sin \theta_p \cdot \sin \phi_q ,$$

$$\Delta f_{p'q'} = \frac{2f_o}{c} \Omega R \sin \theta_{p'} \cdot \sin \phi_{q'} ,$$

and p, p', q, and q' are integers designating each of the divided points on the surface of tne satellite. Equation (24) can then be rewritten as

$$(25)\qquad e_o(t)_{p-p',q-q'} = A(\theta_p \cdot \phi_q)A(\theta_{p'} \cdot \phi_{q'})$$

$$\left[\sum_{i=-\infty}^{\infty} V(t-\Delta T_{p'q'} - i\,\overline{N+M}\; T_{s'p'q'})\right.$$

$$\left.\sum_{n=-\infty}^{\infty} U(t-\Delta T_{p'q'} - nT_{s'p'q'})\right] *$$

$$\left[\sum_{i=-\infty}^{\infty} V(t_1+\Delta T_{pq} + i\,\overline{N+M}\; T_{s'pq} - t)\right.$$

$$\left.\sum_{n=-\infty}^{\infty} U(t_1+\Delta T_{pq} + nT_{s'pq} - t)\right] .$$

Since the component pulses are of very short duration, almost approaching the delta function, and since the envelope function varies at a much slower rate than the function of the component pulses, the forming of the products and the order of convolution in Eq. (25) can be arranged, i.e.,

$$(26)\qquad e_o(t)_{p-p',q-q'} = A(\theta_p \cdot \phi_q)A(\theta_{p'} \cdot \phi_{q'}) \left[\sum_{i=-\infty}^{\infty} V(t-\Delta T_{p'q'} - i\,\overline{N+M}\; T'_{sp'q'})\right.$$

$$\left. * \sum_{i=-\infty}^{\infty} V(t_1+\Delta T_{pq} + i\,\overline{N+M}\; T_{spq} - t)\right] .$$

$$\left[\sum_{n=-\infty}^{\infty} U(t - \Delta T_{p'q'} - nT'_{sp'q'}) * \sum_{n=-\infty}^{\infty} U(t_1 + \Delta T_{pq} + nT'_{spq} - t) \right] .$$

Further, the product of the sums is composed of the sums of the products in Eq. (26); thus Eq. (26) can be rearranged as

$$(27) \qquad e_o(t)_{p-p',q-q'} = A(\theta_p \cdot \phi_q)\, A(\theta_{p'} \cdot \varphi_{q'})$$

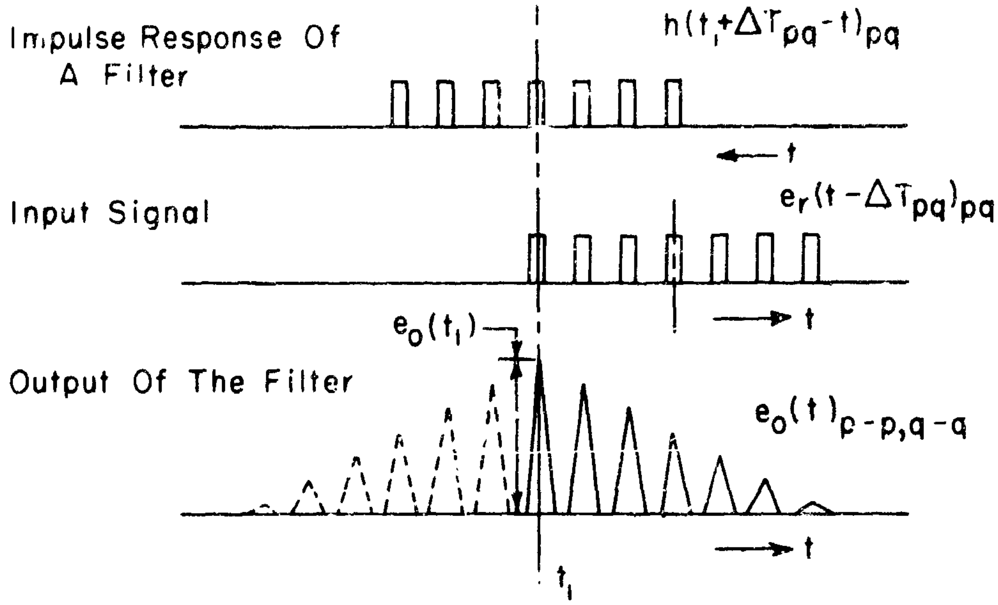
$$\left[\sum_{i=-\infty}^{\infty} V(t - \Delta T_{p'q'} - i\,\overline{N+M}\, T'_{sp'q'}) * \right.$$

$$\left. V(t_1 + \Delta T_{pq} + i\,\overline{N+M}\, T'_{spq} - t) \right] .$$

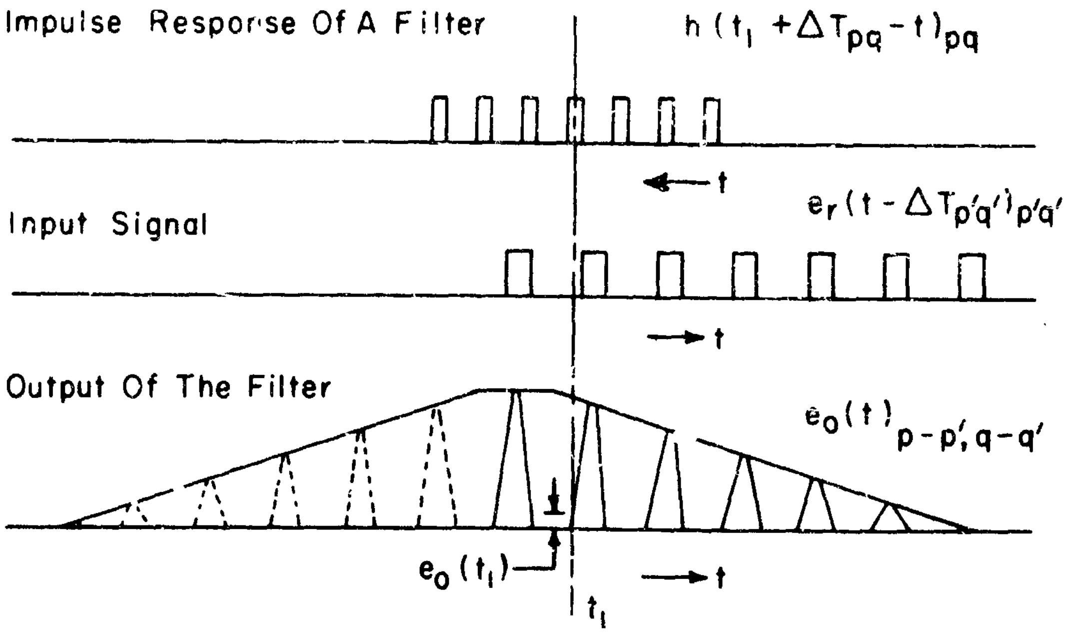
$$\left[\sum_{n=-\infty}^{\infty} U(t - \Delta T_{p'q'} - nT'_{sp'q'}) * \right.$$

$$\left. U(t_1 + \Delta T_{pq} + nT'_{spq} - t) \right] .$$

Figure 6 shows the output waveforms of a matched filter. Figure 6(a) shows a filter output in the case when the response of the filter is matched to the input waveform; Fig. 6(b) shows the same filter output in the mismatched case. In Fig. 6(a) the output of the filter $e_o(t)$ shows a maximum value at observing time t_1, while in Fig. 6(b) it shows a lower value at the same time, t_1. Therefore by observing the response of the filter at a fixed time, t_1, we can discriminate between signals which are or are not matched to the response of the filter. Figures 6(c) and (d) show the output of a filter which is designed for a response to be matched only to the envelope function of the input signal. We can see that the output is maximum $e_o(t_1)$ in Fig. 6(c); however, $e_o(t_1)$ in Fig. 6(d) does not show such a lower value as is shown in (b). From the above estimation we see that it is difficult to distinquish the difference between each input signal by a filter if the response of the filter is designed only to be matched to the envelope characteristics of the input signal. Since it seems to be

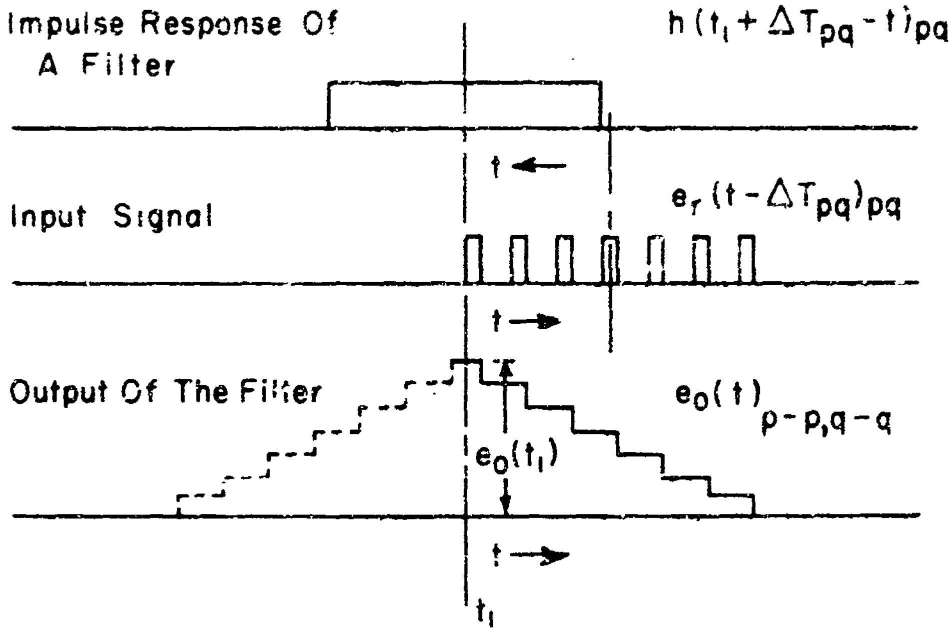


(a) Matched Output Of A Filter



(b) Mismatched Output Of A Filter

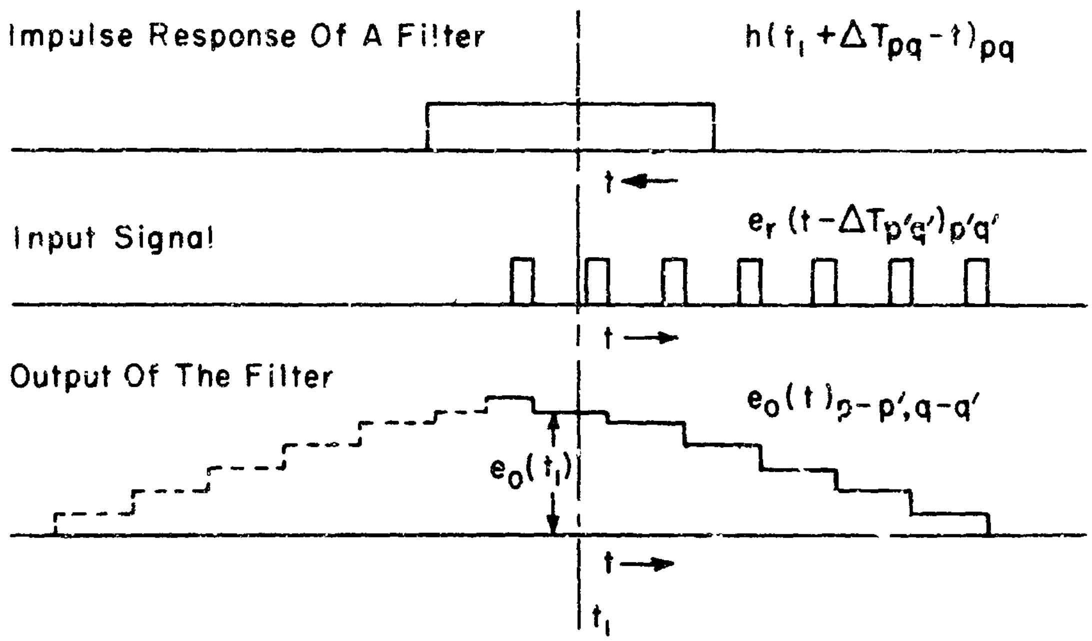
Fig. 6. Outputs of the filters in time domain.



(c) Matched Output Of A Filter

Fig. 6. Outputs of the filters in time domain.

difficult to discriminate the signal in the above case, let us consider the same problem in the frequency domain. If the incident pulse train on the target is finite because of time available for illumination, the spectrum of the received signal becomes a continuous spectrum as shown in Fig. 7 rather than as shown in Fig. 4(g). From the design of the matched filter response such that only the main lobe component in the spectrum of the reflected signal is passed, we can discriminate between matched signal and mismatched signal as shown in Figs. 7(a) and (b), respectively, provided that the deviation by doppler shift between two signals which are reflected from each adjacent portion on the satellite surface is large in comparison with the bandwidth of the main lobe component in the reflected signal. The above discussions will be helpful in understanding the relations between signal and filter response.



(d) Mismatched Output Of A Filter

Fig. 6. Outputs of the filters in time domain.

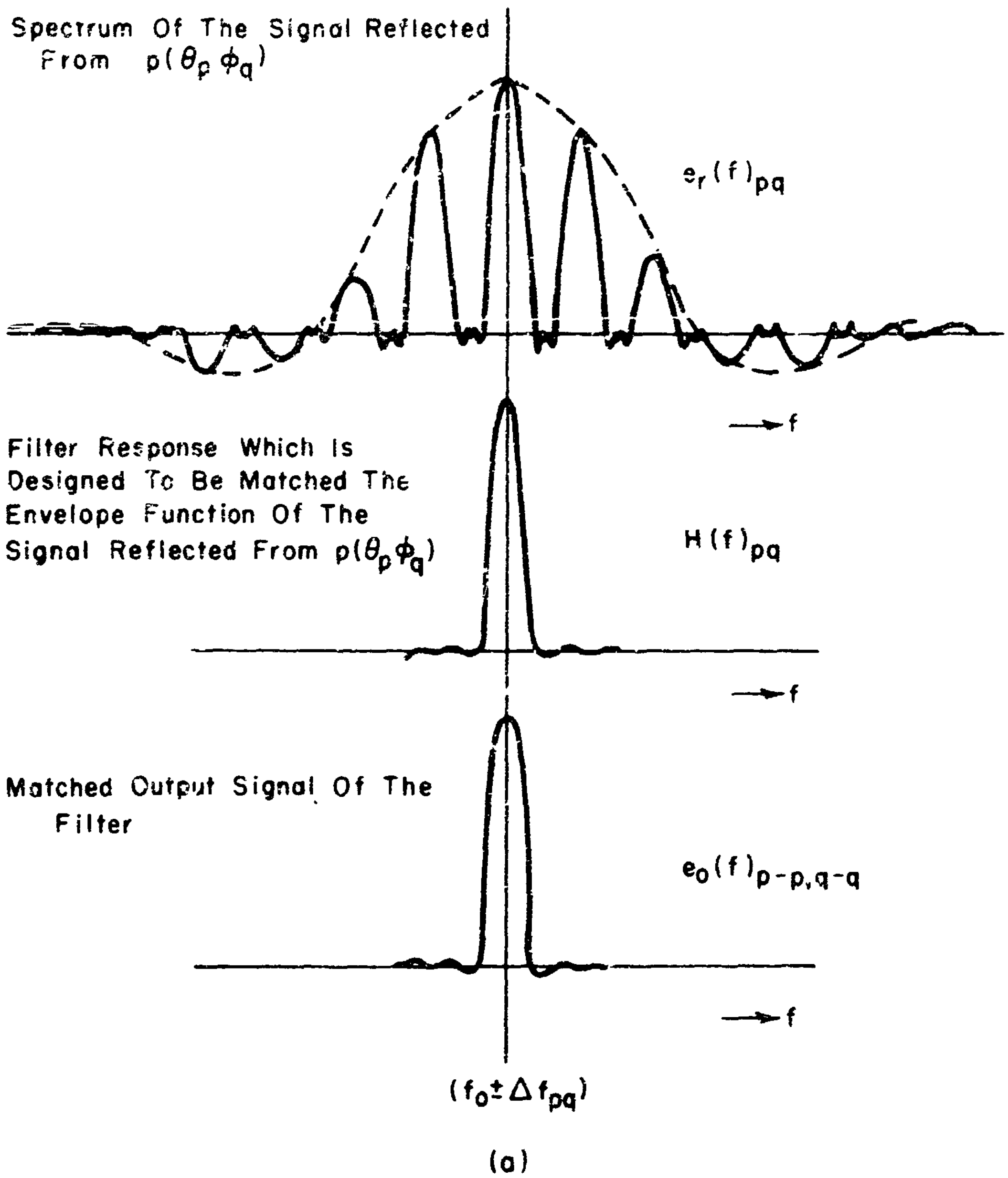


Fig. 7. Outputs of the filters in freq. ency domain.

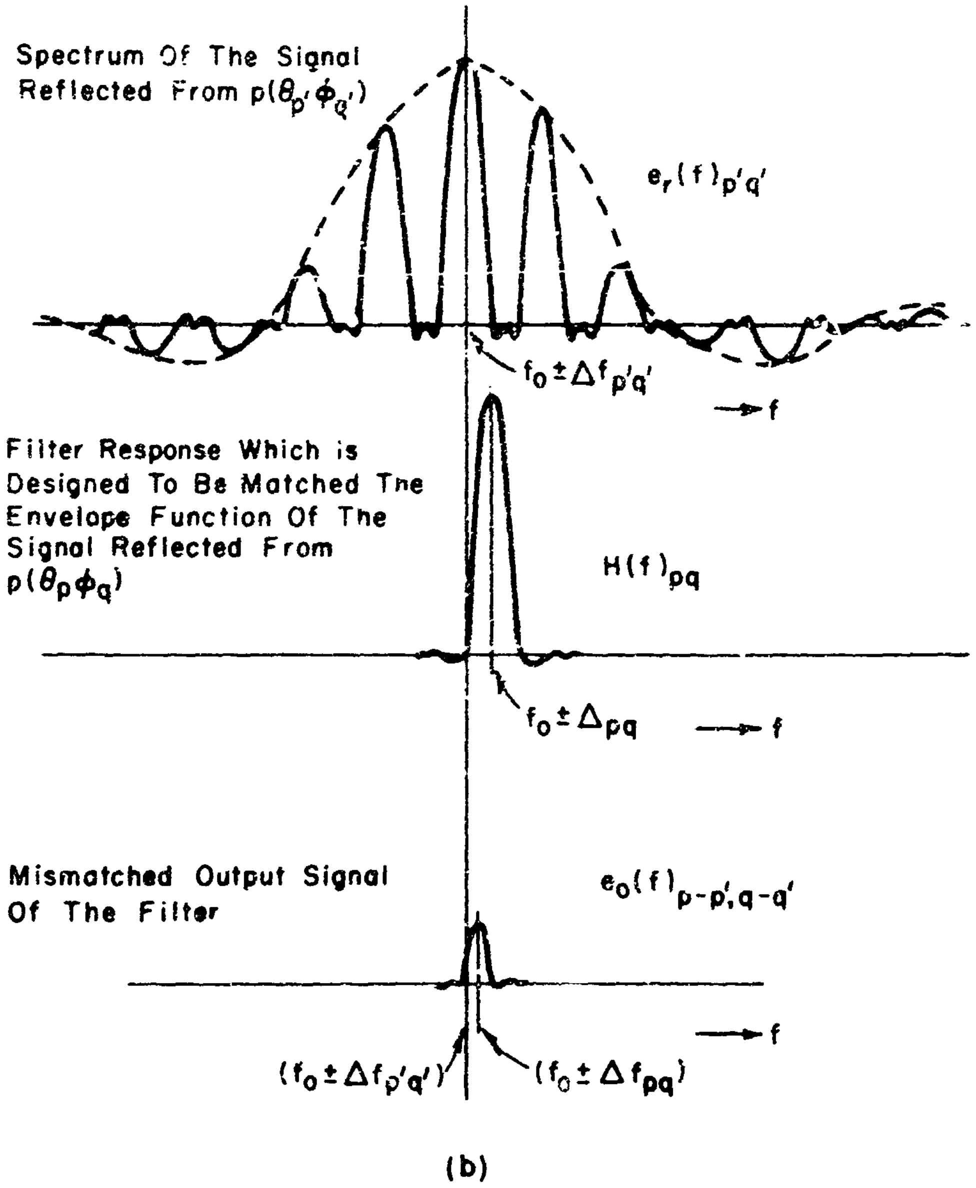


Fig. 7. Outputs of the filters in frequency domain.

CALCULATION OF THE AMBIGUITY FUNCTION

We may write the ambiguity function of the reflected waveform of a point of the satellite surface as $|X(\Delta f, \Delta T, f_d, T_d)|$ rather than the usual expression of $|X(f_d, T_d)|$.

From the definition of the ambiguity function,

$$(28) \qquad |X(\Delta f, \Delta T, f_d, T_d)| = \left| \int_{-\infty}^{\infty} e_r(t-\Delta T) \cdot \widetilde{e_r}(t-\Delta T - T_d) \, \epsilon^{-j2\pi(f_d \pm \Delta f)t} dt \right| ,$$

where

$\widetilde{e_r}(t)$ is the complex conjugate of $e_r(t)$ and

$$\pm \Delta f \begin{cases} -\Delta f \text{ at } 0 < \phi \leqq \frac{\pi}{2} \\ +\Delta f \text{ at } 0 > \phi \geqq \frac{\pi}{2} \end{cases} .$$

By substituting Eq. (15) into Eq. (28)

$$(29) \qquad |X(\Delta f, \Delta T, f_d, T_d)| = \left| \int_{-\infty}^{\infty} A(\theta,\phi) \sum_{i=-\infty}^{\infty} V(t-\Delta T - i\,\overline{N+M}\,T_s') \sum_{n=-\infty}^{\infty} U(t-\Delta T - nT_s') \cdot \widetilde{A}(\theta,\phi) \cdot \sum_{i=-\infty}^{\infty} \widetilde{V}(t-\Delta T - T_d - i\,\overline{N+M}\,T_s') \cdot \sum_{n=-\infty}^{\infty} \widetilde{U}(t-\Delta T - T_d - nT_s') \cdot \epsilon^{-j2\pi(f_d \pm \Delta f)t} dt \right| .$$

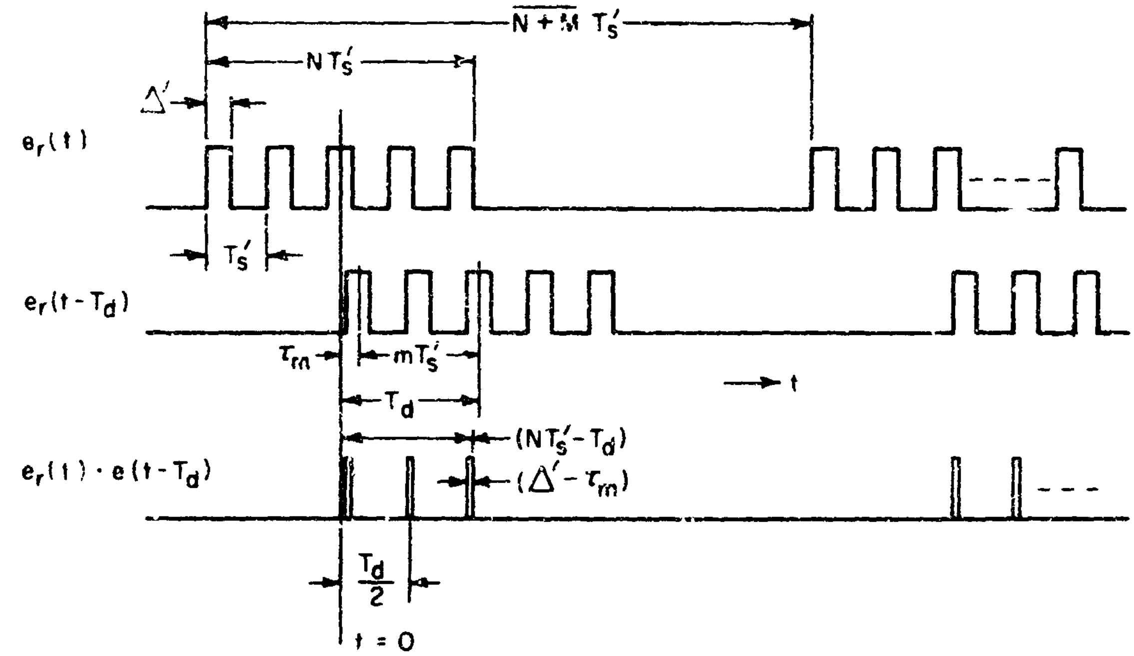
If we select the coordinate origin at $(\Delta f, \Delta T)$, Eq. (29) becomes $|X(0, 0, f_d, T_d)|$ and

$$(30) \qquad |X(0,0,T_d,f_d)|^2 = A^2(\theta,\phi)\cdot \left| \int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} V(t-i\,\overline{N+M}\,T_s') \cdot \right.$$

$$\sum_{i=-\infty}^{\infty} \tilde{V}(t-T_d-i\,\overline{N+M}\,T_s') \cdot \sum_{n=-\infty}^{\infty} U(t-nT_s') \cdot$$

$$\left. \sum_{n=-\infty}^{\infty} \tilde{U}(t-T_d-nT_s') \cdot \epsilon^{-j2\pi f_d t} dt \right| .$$

Now, let us consider the problem of integral Eq. (30). As shown in Fig. 8(a), the function to be integrated in Eq. (30) is a pulse train which is produced by the multiplication of a pulse train with the same but delayed one. The resultant pulse consists of the envelope function $V'(t)$ and component pulse function $U'(t)$. As shown in Fig. 8(b), $V'(t)$ has the width of



(a)

Fig. 8. Waveforms of the function which are integrant in Eq. (30).

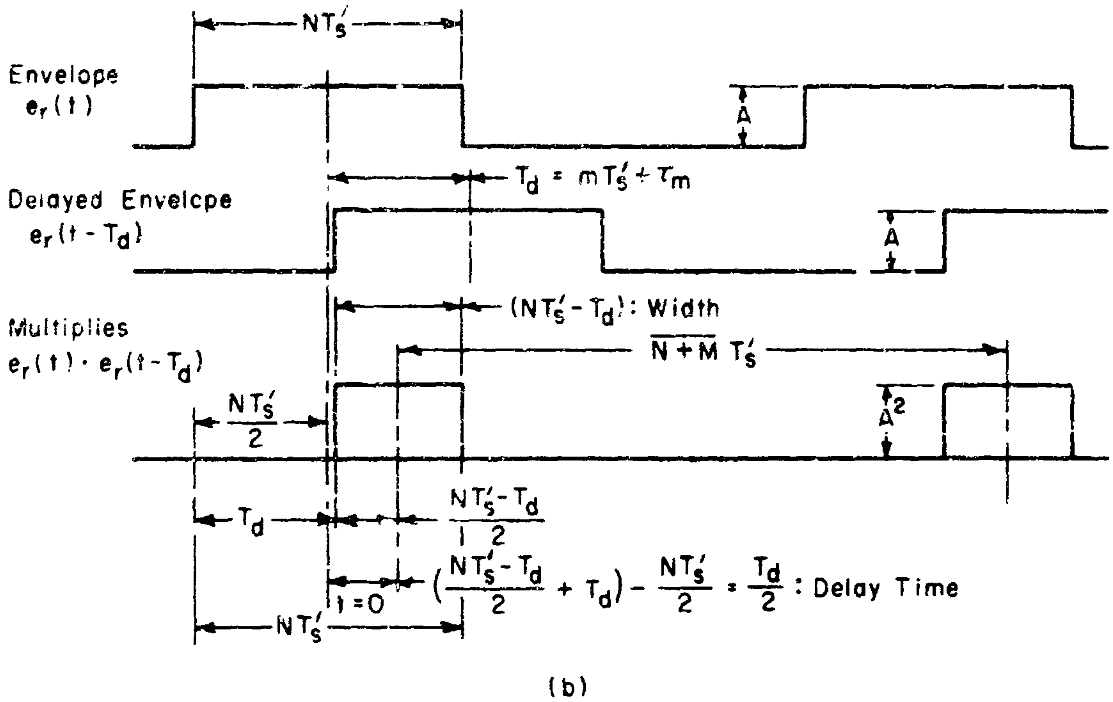


Fig. 8. Waveforms of the function which are integrant in Eq. (30).

$(NT_s' - T_d)$, delay time of $T_d/2$, and repetition period $\overline{N+M}\, T_s'$. From similar estimation of $U'(t)$, we know that it has the width of $(\Delta' - \tau_m)$, delay time of $\tau_m/2$, and repetition period of T_s'. Thus we can rewrite the function to be integrated in Eq. (30) as

$$(31) \qquad \sum_{i=-\infty}^{\infty} V(t - i\,\overline{N+M}\, T_s') \cdot \sum_{i=-\infty}^{\infty} \tilde{V}(t - T_d - i\,\overline{N+M}\, T_s') \cdot$$

$$\sum_{n=-\infty}^{\infty} U(t - nT_s') \cdot \sum_{n=-\infty}^{\infty} \tilde{U}(t - T_d - nT_s') = \sum_{i=-\infty}^{\infty} V'\left(t - \frac{T_d}{2} - i\,\overline{N+M}\, T_s'\right) \cdot$$

$$\sum_{n=-\infty}^{\infty} U'\left(t - \frac{\tau_m}{2} - nT_s'\right),$$

where

$$\tau_d = mT_s' + \tau_m \text{ and } \begin{cases} m = 1, 2, 3, \text{--------} \\ -\frac{\Delta'}{2} \leq \tau_m \leq \frac{\Delta'}{2} \end{cases} .$$

Equation (30) becomes

$$(32) \qquad |X(0,0,f_d,T_d)| = A^2(\theta,\phi) \left| \int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} V'(t - \frac{T_d}{2} - i\,\overline{N+M}\,T_s') \cdot \right.$$

$$\left. \sum_{n=-\infty}^{\infty} U'(t - \frac{\tau_m}{2} - nT_s')\epsilon^{-j2\pi f_d t} dt \right| .$$

By convolution

$$(33) \qquad |X(0,0,f_d,T_d)| = A^2(\theta,\phi) \left\| \left[\int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} V'(t - \frac{T_d}{2} - i\,\overline{N+M}\,T_s') \cdot \right. \right.$$

$$\left. \epsilon^{-j2\pi f_d t} dt \right] * \left[\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} U'(t - \frac{\tau_m}{2} - nT_s') \cdot \right.$$

$$\left. \left. \epsilon^{-j2\pi f_d t} dt \right] \right\| ,$$

$$= A^2(\theta,\phi) \left| \left[\sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} V'(t - \frac{T_d}{2} - i\,\overline{N+M}\,T_s') \cdot \epsilon^{-j2\pi f_d t} dt \right] * \right.$$

$$\left. \left[\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} U'(t - \frac{\tau_m}{2} - nT_s') \cdot \epsilon^{-j2\pi f_d t} dt \right] \right| .$$

The Fourier transform of the envelope $V'(t)$ is

$$(34) \qquad \int_{-\infty}^{\infty} V'\left(t - \frac{T_d}{2} - i\,\overline{N+M}\,T_s'\right) \cdot \epsilon^{-j2\pi f_d t} dt$$

$$= \left[\frac{2}{\overline{N+M}\,T_s'} \int_{-\frac{NT_s'-T_d}{2}}^{\frac{NT_s'-T_d}{2}} \epsilon^{-j2\pi\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)\left(t + \frac{T_d}{2}\right)} dt\right]^2$$

$$= \left[\frac{2}{\overline{N+M}\,T_s'} \cdot \frac{\sin^2\pi\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)(NT_s' - T_d)}{\pi^2\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)^2} \cdot \epsilon^{-j2\pi\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)\frac{T_d}{2}}\right]^2,$$

$$= \left(\frac{2}{\overline{N+M}\,T_s'}\right)^2 \frac{\sin^2\pi\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)(NT_s' - T_d)}{\pi^2\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)^2} \cdot \epsilon^{-j2\pi\left(f_d - \frac{i}{\overline{N+M}\,T_s'}\right)\frac{T_d}{2}}.$$

The Fourier transform of the component pulse function $U'(t)$ is

$$(35) \qquad \int_{-\infty}^{\infty} U'\left(t - \frac{\tau_m}{2} - nT_s'\right) \epsilon^{-j2\pi f_d t} dt = \left[\frac{2}{T_s'} \int_{-\frac{\Delta'-\tau_m}{2}}^{\frac{\Delta'-\tau_m}{2}} \epsilon^{-j2\pi\left(f_d - \frac{n}{T_s'}\right)\left(t + \frac{\tau_m}{2}\right)} dt\right]^2,$$

$$= \left(\frac{2}{T_s'}\right)^2 \frac{\sin^2 \pi \left(f_d - \frac{n}{T_s'}\right)(\Delta' - \tau_m)}{\pi^2 \left(f_d - \frac{n}{T_s'}\right)^2} \cdot \epsilon^{-j2\pi \left(f_d - \frac{n}{T_s'}\right)\frac{\tau_m}{2}} .$$

By substituting Eqs. (34) and (35) into Eq. (33) we obtain

$$(36) \qquad |X(0,0,f_d,T_d)| = A^2(\theta,\phi) \cdot \left(\frac{2}{N+M\,T_s'} \cdot \frac{2}{T_s'}\right)^2 \cdot$$

$$\left| \left[\sum_{i=-\infty}^{\infty} \frac{\sin^2\pi\left(f_d - \frac{i}{N+M\,T_s'}\right)(NT_s' - T_d)}{\pi^2\left(f_d - \frac{i}{N+M\,T_s'}\right)^2} \cdot \epsilon^{-j2\pi\left(f_d - \frac{i}{N+M\,T_s'}\right)\frac{T_d}{2}} \right] * \right.$$

$$\left. \left[\sum_{n=-\infty}^{\infty} \frac{\sin^2\pi\left(f_d - \frac{n}{T_s'}\right)(\Delta' - \tau_m)}{\pi^2\left(f_d - \frac{n}{T_s'}\right)} \cdot \epsilon^{-j2\pi\left(f_d - \frac{n}{T_s'}\right)\frac{\tau_m}{2}} \right] \right| .$$

Replacing f_d and $T_d/2$ by $f_d \pm \Delta f$ and $\frac{T_d + \Delta T}{2}$, respectively, we obtain a formula for the ambiguity functio- which includes doppler shift Δf and time delay ΔT;

$$(37) \qquad |X(\Delta f,\Delta T,f_d,T_d)| = \left(\frac{4A(\theta,\phi)}{N+M\,T_s' \cdot T_s'}\right)^2$$

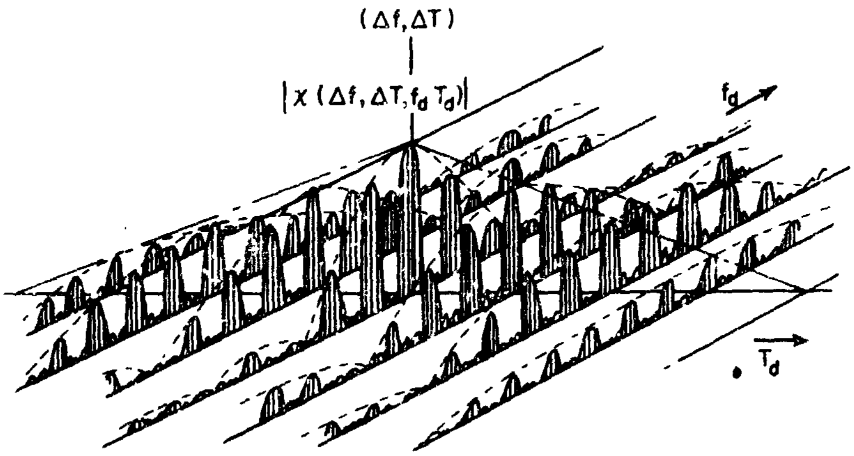
$$\left| \left[\sum_{i=-\infty}^{\infty} \frac{\sin^2\pi\left(f_d - \frac{i}{N+M\,T_s'} \pm \Delta f\right)(NT_s' - T_d)}{\pi^2\left(f_d - \frac{i}{N+M\,T_s'} \pm \Delta f\right)} \cdot \right. \right.$$

$$\left. \epsilon^{-j\pi\left(f_d - \frac{i}{N+M\,T_s'} \pm \Delta f\right)(T_d + \Delta T)} \right] *$$

$$\left[\sum_{n=-\infty}^{\infty} \frac{\sin^2 \pi \left(f_d - \frac{n}{T_s'} \pm \Delta f \right)(\Delta' - \tau_m)}{\left(f_d - \frac{n}{T_s'} \pm \Delta f \right)} \cdot \right.$$

$$\left. \epsilon^{-j\pi \left(f_d - \frac{n}{T_s'} \pm \Delta f \right)(\tau_m + \Delta T)} \right] \Bigg| \cdot$$

Figures 9 (a), (b), and (c) show the diagram of the ambiguity function of Eq. (37).



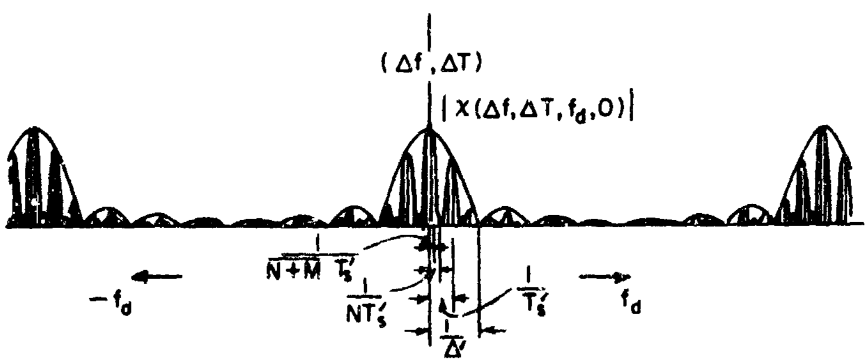
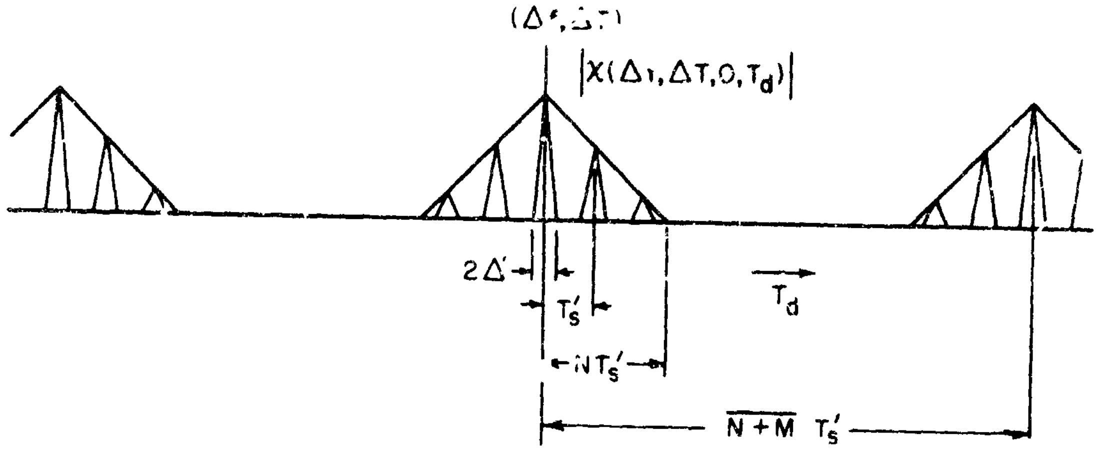


Fig. 9. The ambiguity diagrams of a matched filter output. (a) on both axes, f_d and T_d, (b) on the f_d axis.



(c)

Fig. 9. The ambiguity diagrams of a matched filter output. (c) on the T_d axis.

CONCLUSIONS

If the transmitted radar signal is a pulse train as shown in Fig. 3, the reflected wave from a point $P(\theta, \phi)$ on the surface has the figure of discrete frequency spectrum as shown in Fig. 4, the formula for which is given in Eq. (18).

From the interpretation of Fig. 4, we can see that the origin of the coordinate in Fig. 4 corresponds to the carrier frequency with doppler shift $\pm \Delta f$. The amplitude of the spectrum at the origin has $A(\theta, \phi) \cdot 2\beta_\ell$. The spectral lines are separated by an interval of $1/\overline{N+M}\, T_s'$.

The intervals in the spectrum between the main lobe and the first side lobe, and between the side lobes are $1/T_s'$. The width of the main lobe is $2/NT_s'$, which depends on the surface point $P(\theta, \phi)$. The width of the envelope of the positive lobes about the origin is $2/\Delta'$, which also depends on $P(\theta, \phi)$.

We then know that all of the above parameters are some functions of doppler shift Δf and delay time ΔT at reflection point $P(\theta, \phi)$.

The transfer function and impulse response of the desired matched filter are shown in Eqs. (20) and (22).

Figure 7 shows the output of the matched filter $H(f)_{pq}$ against the signal $e_r(f)_{p'q'}$ reflected from a point $P(\theta_{p'}, \phi_{q'})$ of the surface. Figure 9 shows the ambiguity diagram of the signal reflected from a point $P(\theta, \phi)$. We can see that it resembles the discrete figure of Fig. 1 and all parameters also become some functions of Δf and ΔT.

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